The Path Integral, Wilson Lines, and Disorder Operators II

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Abstract

This lecture is intended to follow Phil's talk on "Categorical Geometric Langlands and Quantum Field Theory" and its relevance to the Langlands program. We start with a review of field theory, extending the ideas of classical field theory to the path integral formulation of quantum field theory (QFT). In particular we study the various physical "observables" that arise out of the path integral, introducing the operator product expansion and Wilson loops. Finally, we discuss *disorder operators*, whose insertion into the path integral changes the space of fields that we integrate over to include singularities. For this, we study 't Hooft lines via the picture of monopoles in 3D.

1 Recall of Last Lecture

Remember that for a given classical field theory consisting of

- A *spacetime manifold* M
- A space of sections ("fields") a fiber bundle $E \to M$.
- An action $S[\Phi]$ from the space of field configurations into $\mathbb{C}$.

we defined the **partition function** of our quantum field theory to be

$$\mathcal{Z} = \int \mathcal{D}\Phi \, e^{-S[\Phi]}. \tag{1}$$

Where this is is a functional integral over the space of all fields known as the **Feynman Path Integral**. The measure on this space is mathematically ill-defined in general.

Definition 1 (Classical Observable)**.** A classical observable is a functional from the set of field configurations to the ground field $\mathbb{C}$.

Definition 2 (Observable)**.** A **quantum observable** (which we will refer to as just an *observable* in these lectures) is a functional from the a field theory into the ground field $\mathbb{C}$. In the Feynman picture, it can be seen as a statistical average of classical observables over all field configurations.

In the last lecture we began to focus on *gauge theory*, namely when X is an associated bundle to a G-bundle for G a reductive Lie group.

There, we defined the corresponding operator

$$W_R(\gamma) := \operatorname{Tr} R(\operatorname{Hol}(A, \gamma)) \tag{2}$$

for γ a closed curve. We saw that for $\gamma \to \gamma'$ the operator product expansion gives us that

$$\lim_{\gamma \to \gamma'} W_R(\gamma) W_{R'}(\gamma') = \sum_\alpha n_\alpha W_{R_\alpha}(L') \tag{3}$$

where n_α is the multiplicity with which the representation R_α appears in $R \otimes R'$.

2 Montonen-Olive Duality

We will be working in 4D $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory. This theory takes a little bit of time to describe in detail, but the fields of primary interest to us in this theory will be a connection 1-form A known as the "gauge field" and its curvature form F together with an ad-$\mathfrak{g}$ valued 1-form ϕ. These play distinct roles in the discussion that follows.

With the stage set, this is now our idea:

Concept 3 (Montonen-Olive Duality)**.** In 4D $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group G and complex coupling constant τ, any correlator of observables

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\tau,G} := \int \mathcal{D}\{Fields\}\, \mathcal{O}_1 \ldots \mathcal{O}_n\, e^{-S}$$

can be rewritten in terms of Yang-Mills theory with inverse coupling constant $-1/\tau$ on the Langlands dual group $\check{G}$ as a correlator of dual operators $\tilde{\mathcal{O}}_1 \ldots \tilde{\mathcal{O}}_n$

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\tau,G} = \left\langle \tilde{\mathcal{O}}_1 \ldots \mathcal{O}_n \right\rangle_{-1/\tau,\check{G}}.$$

In particular $\mathcal{N} = 4$ super Yang-Mills theory has a $\mathbb{CP}^1$ family of topological twists. Two of these will be relevant here, known as the $\hat{A}$-model and the $\hat{B}$-model[1]. This twisting introduces an asymmetry between G and $\check{G}$.

In the $\check{G}$ theory: the $\hat{B}$ model, A and ϕ combine into a (complex-valued) connection $\mathcal{A} = A + i\phi$. The equations of motion together with supersymmetry requires $\mathcal{A}$ to be flat in the $\hat{B}$ model. As long as this flat connection is irreducible, it is the only relevant variable in the $\hat{B}$.

Now let $L \subset M$ be an oriented 1-manifold embedded in M. On the $\hat{B}$-model side, we can consider taking the holonomy of the connection $\mathcal{A}$ along L, when L is closed, giving us a Wilson loop. The $\hat{B}$ model condition on the flatness of $\mathcal{A}$ implies that the holonomy of the Wilson loop only depends on the homotopy class of L

If M has boundaries, we can let L be an open 1-manifold connecting two ends of M. Then, the Wilson operator will give us matrix elements between the initial and final states of the theory. Because Wilson operators geometrize $\mathrm{Rep}(\check{G})$, the space of physical states living on the boundary of M is exactly $\check{R}$ for some $\check{R} \in \mathrm{Rep}(\check{G})$. A Wilson loop connecting boundary components gives us a matrix element between initial and final vectors in $\check{R}$.

In the G theory: the $\hat{A}$ model, A and ϕ instead obey a different equation:

$$F - \phi \wedge \phi = \star D_A \phi. \tag{4}$$

This equation is analogous to the equation of motion for the 2D A models. We will see how the Bogomolny equations for magnetic monopoles arise as a special restriction of this equation in the next section.

From the above discussion, we should ask

Question. What is the dual operator to a Wilson line?

From the physics viewpoint, 't Hooft showed in the 1980s that MO duality will exchange a Wilson line (a type of "order operator") on one side with something known as a 't Hooft line (a type of "disorder operator") on the other side.

[1]This notation comes from the fact that, upon compactification down to two dimensions, these models become the A and B topological sigma models discussed before

As we mentioned last class, we can understand the insertions of ‘t Hooft lines in the path integral as imposing divergence conditions on the curvature form F so that in local coordinates $x^1 \dots x^3$ perpendicular to the line we have

$$F(\vec{x}) \sim \star_3 d\left(\frac{\mu}{2r}\right) \tag{5}$$

where μ is an element of the lie algebra $\mathfrak{g}$. It turns out that for us to be able to find a gauge field A whose curvature F satisfies this condition, we must have that μ is a Lie algebra homomorphism $\mathbb{R} \to \mathfrak{g}$ obtained as the pushforward of a Lie group homomorphism $U(1) \to G$.

Another way to say this is (after using gauge freedom to conjugate μ to a particular Cartan subalgebra) that μ must lie in the coweight lattice Λ_{cw}. Note though, that if we perform a gauge transformation by

$$\exp(i\pi(E_\alpha + E_{-\alpha})/\sqrt{2\alpha^2})$$

this will send

$$\mu \to \mu - 2\alpha\alpha \cdot \mu / \langle \alpha, \alpha \rangle$$

which corresponds to a Weyl group action on μ. This turn out to be the only degeneracy, so we have that ‘t Hooft operators are classified by the space:

$$\Lambda_{cw}(G)/\mathcal{W}.$$

But this is also the same as

$$\Lambda_w(\check{G})/\mathcal{W}.$$

We know that this is the space of representations of the Langlands dual group.

Proposition 4. *‘t Hooft operators in gauge group G are classified by irreducible representations of $\check{G}$.*

The operator product expansion of Wilson lines captures the monoidal category structure of $\mathrm{Rep}(\check{G})$. By duality, this category must also be capturing the OPE of ‘t Hooft lines. Can we say anything about the OPE of ‘t Hooft lines in terms of G?

3 Operator Product Expansion of ‘t Hooft Lines

Because the operator product expansion is a local process, we can assume our base manifold looks like anything. It turns out to be fruitful to take $X = I \times C \times \mathbb{R}$. Here, I is the unit interval $(0, 1)$, C is a Riemann surface (which we can take to be $\mathbb{CP}^1$ WLOG) and $\mathbb{R}$ is regarded as the “time” direction and adopt a Hamiltonian point of view on $W = I \times C$.

The boundary conditions on I matter here, and it turns out that in the $\hat{A}$ model we should consider *Dirichlet* boundary conditions on one end and *Neumann* boundary conditions on the other. In the language of gauge theory, Dirichlet boundary conditions demand the bundle to be trivial on that boundary, while Neumann boundary conditions allow for it to be arbitrary.

Now ‘t Hooft lines look like points on the 3-manifold $W = I \times C$. We can locally take $\phi = \phi_4 dx^4$ so that on W, ϕ behaves as a scalar. Then, on W, Equation (5) reduces to the **Bogomolny equations** for monopoles:

$$F = \star_3 D_A \phi.$$

Let's write a local coordinate $z \in \mathbb{C}$ parameterizing C and $\sigma \in \mathbb{R}$ parameterizing I. Gauging away $A_\sigma = 0$, these equations reduce to the following:

$$\partial_\sigma A_{\bar{z}} = -iD_{\bar{z}}\phi.$$

This condition can be interpreted as stating that the isomorphism class of the holomorphic G-bundle corresponding to the connection $A_{\bar{z}}$ is independent of y. This is because the right hand side corresponds to changing A by a gauge transformation generated by $-i\phi$. Thus, gauge transforming $A \to A + i\phi$ gives us a holomorphic connection on the new G-bundle, putting it in the same holomorphic class.

The only place where this is violated is at the values σ where the Bogomolny equations become singular. This is where we have the insertion of a 't Hooft operator.

More explicitly, the Langlands dual is defined so that any highest weight representation $\hat{\rho} : \hat{G} \to U(1)$ is dual to a morphism $\rho : U(1) \to G$ which can be viewed as a *clutching function* for a G bundle on the Riemann sphere $\mathbb{CP}^1$. Complexifying this gives $\rho : G \to \mathbb{C}^\times \cong \mathbb{CP}^1 \backslash \{p, q\}$, AKA gluing a trivial bundle over $\mathbb{CP}^1 \backslash \{p\}$ to a trivial bundle over $\mathbb{CP}^1 \backslash \{q\}$. This is exactly what we call a Hecke modification of type ρ. Every holomorphic G-bundle over $\mathbb{CP}^1$ arises in this way. Phil has taught us before that we should recognize the space of Hecke modifications as the affine Grassmannian $Gr_G = G((z))/G[[z]]$

It turns out that for $\mathcal{N} = 4$ supersymmetric Yang Mills, the space of physical states is the (intersection) cohomology of the space of solutions to the Bogomolny equations with prescribed singularities labeled by $\check{R}_i, p_i$[2]. We denote this space by $\mathcal{Z}(\check{R}_1, p_1, \ldots, \check{R}_k, p_k)$. Because the underlying field theory is topological, and because the space of n-tuples on W is simply connected (so no monodromy can occur), we have that $\mathcal{Z}$ does not depend on the explicit positions of any of the p_i. Thus we can write $\mathcal{H}(\check{R}_1, \ldots \check{R}_k) = H^*(\mathcal{Z})$ and define this as the *space of physical states* for this given set of line defect insertions.

Further, $\mathcal{Z}(\check{R}_1, \ldots \check{R}_k)$ turns out to topologically be a simple product $\prod_{i=1}^k \mathcal{Z}(\check{R}_i)$ where $\mathcal{Z}(\check{R}_i)$ is the same as the compactified space $\mathcal{N}(\check{R}_i)$ of Hecke modifications of type $\check{R}_i$, then by using the fact that *the product of cohomologies is the cohomology of the product* we obtain:

$$\mathcal{H}(\check{R}_1, \ldots \check{R}_k) = \bigotimes_{i=1}^{k} \mathcal{H}(\check{R}_i) \tag{6}$$

This suggests that there is an isomorphism of $\check{R}_i$ and $\mathcal{H}(\check{R}_i)$ as vector spaces. Indeed, it can be shown that such an isomorphism is the only way for these categories of (finite dimensional) vector spaces to have the same monoidal structure.

I might try to write more about this or expand on the previous two paragraphs

4 The Action of Wilson Loops on Boundary Conditions (time permitting)

If we assume that $M = \Sigma \times C$ for C a compact Riemann surface and Σ a (not necessarily compact) surface with boundary, we can study loop insertions more naturally. The following is a simplified picture of the general case:

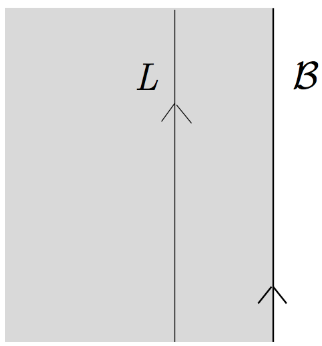
[2] In general, there are so-called "instanton corrections" to this space of states, but they are absent in this situation for reasons relating to supersymmetry.

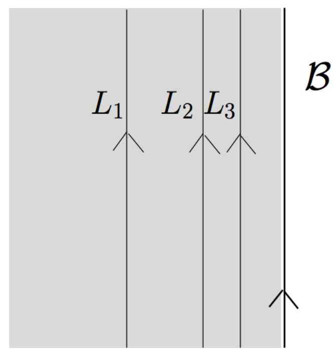
Definition 5 (Hitchin's Moduli Space)**.** $\mathcal{M}_H(G, C)$ is the space of solutions to the Hitchin equations on a curve C.

If we consider C to be "small" relative to Σ, for each point in Σ, the additional data for the field configurations on the space C must give us a point in this moduli space. That is, we get a nonlinear sigma model on $\Sigma \to \mathcal{M}(G, C)$.

Let the curve defining a (Wilson or 't Hooft) operator be $\gamma = \gamma_0 \times p$ in $\Sigma \times C$ with p a point on C and γ_0 a curve on Σ. Let $\partial\Sigma_0$ be a connected component of $\partial\Sigma$. A boundary condition for the field theory on Σ_0 is called a **brane**.

Let γ_0 approach this boundary. On the $\hat{B}$ side, the insertion of a Wilson loop acts as an associative endofunctor for the category of boundary conditions on the topological sigma model on Σ with target $\mathcal{M}_H(G, C)$. This target space, with choice of complex structure J, can be identified with $\mathcal{M}_{flat}(G, C)$.





This functor will depend on the point $p \in C$ corresponding to the Wilson line. Consider the product $\mathcal{M}_{flat}(G, C) \times C$. There is a universal G-bundle $\mathcal{E}$ over this space, given by taking a point in $\mathcal{M}_{flat}$ and restricting the corresponding bundle to a point in C.

Given any coherent sheaf on $\mathcal{M}_{flat}$, we can tensor this with $R(\mathcal{E})$. This is the action of the Wilson loop insertion on the space.

Consider the structure sheaf $\mathcal{O}_x$ of a point $x \in \mathcal{M}_{flat}(\check{G}, C)$. For any representation $\check{R}$, the Wilson loop maps $\mathcal{O}_x$ to $\mathcal{O}_x \otimes \check{R}$. Thus $\mathcal{O}_x$ is an eigenobject for the functor $W_{\check{R}}(p)$, which acts on it by tensoring it with the vector space $\check{R}(\mathcal{E}_p)_x$. In fact, letting p vary we see that it is an eigenobject for all $W_{\check{R}}(p)$. Another way of saying this is that the eigenvalue is the flat $\check{G}$-bundle $\check{R}(\mathcal{E})_x$ on C.

More directly, this flat bundle is obtained by taking the flat principle bundle on C corresponding to x and forming the associated bundle via $\check{R}$.

The action of the 't Hooft operators is more difficult to see. They will end up acting by Hecke transformations on the space of boundary conditions. By Monotonen-Olive duality, it turns out that the brane corresponding to a fiber of the Hitchin fibration in $\mathcal{M}_H(G, C)$ is a common eigenobject for all operators.